Winter term 2026/27
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Presence sheet 3

Mathematics for Machine Learning

Tutorial of **Week 4 (02. - 06.11.2026)**

Exercise 1 (**Scalar product).**
Consider the vector space $V = \mathcal{C}([a,b])$ for $a < b$. Prove that the following function $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ with $\langle f, g \rangle = \int_a^b f(t)g(t)\mathrm{d}t$ defines a scalar product.

Exercise 2 (**Orthonormal basis).**
Decide if the following function sets of vectors are orthonormal bases of $\mathbb{R}^3$ with respect to the standard scalar product:

1. $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\} \subset \mathbb{R}^3$
2. $\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ -2 \\ -2 \end{pmatrix} \right\} \subset \mathbb{R}^3$
3. $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \sqrt{2}/2 \\ \sqrt{2}/2 \end{pmatrix}, \begin{pmatrix} 0 \\ \sqrt{2}/2 \\ -\sqrt{2}/2 \end{pmatrix} \right\} \subset \mathbb{R}^3$

Exercise 3 (**Projection).**
Decide if the following matrices are projections.

1. $A = \begin{pmatrix} 0 & 0 \\ 7 & 1 \end{pmatrix}$
2. $B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
3. $C = \begin{pmatrix} 1 & 0 \\ 5 & 4 \end{pmatrix}$

Which of these is an orthogonal projection?

Exercise 4 (**Orthogonal matrix).**
Consider an orthogonal matrix Q. Prove the following properties for the standard scalar product:

1. $\langle Qv, Qw \rangle = \langle v, w \rangle$
2. $\|Qv\| = \|v\|$.
3. $|\det(Q)| = 1$.

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Exercise 5 (**Diagonalizable matrices).**
Find a matrix over $\mathbb{C}$ which is not diagonalizable.

Exercise 6 (**Positive definite matrices).**
Consider the matrix $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & -6 \end{pmatrix}$.

1. Find the eigenvalues of A and their algebraic multiplicity.
2. Decide whether A is invertible.
3. Decide whether A is positive (semi-)definite.