

Presence sheet 02

Mathematics for Machine Learning

Tutorial of **Week 03 (26. - 30.10.2026)**

Exercise 1 (Kernel and image).

Consider the following matrix

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 3 \end{pmatrix} \quad (1)$$

which describes a linear map $T : \mathbb{R}^4 \rightarrow \mathbb{R}^3$ by $T(x) = Ax$.

- a) Determine the rank of A .
- b) Determine the dimension of the kernel of A .
- c) Determine a basis of the image and a basis of the kernel of A .

Exercise 2 (Invertibility).

Is the following matrix invertible?

$$A = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}. \quad (2)$$

Exercise 3 (Determinant).

Calculate the determinant of the following matrices

a) $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}.$

b) $B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 9 \end{pmatrix}.$

c) $C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 5 & 4 \\ 0 & 8 & 9 \end{pmatrix}.$

Exercise 4 (Eigenvalues). Find a (non-diagonal) matrix $A \in \mathbb{R}^{2 \times 2}$ with exactly one (real) eigenvalue.

Exercise 5 (Rotation matrix). Consider the rotation matrix $A = \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix}.$

Show that A has no real eigenvalues for $\alpha \notin \pi\mathbb{Z}$. Try to understand where the name "rotation matrix" comes from!

Exercise 6 (Eigenvectors).

Consider the matrix $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$.

- a) Find one eigenvector of A for eigenvalue $\lambda_1 = 3$.
- b) Find one eigenvector of A for eigenvalue $\lambda_2 = 1$.
- c) Are there more than one eigenvector for each eigenvalue?
- d) Are there more eigenvalues?

Exercise 7 (Eigenspaces).

Consider the matrix $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$.

- a) Calculate the eigenspace of A for eigenvalue $\lambda_1 = 1$, that is, $E(1, A)$.
- b) Calculate the eigenspace of A for eigenvalue $\lambda_2 = 2$, that is, $E(2, A)$.
- c) Find a different basis of $E(1, A)$ and $E(2, A)$.
- d) What is the geometric multiplicity of λ_1 and λ_2

Exercise 8 (Inverse matrices).

Assume you take “just any” 3x3 matrix. Intuitively, what do you think is more “likely”: that it is invertible or not? Discuss informally.