Winter term 2026/27
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Assignment 3

Mathematics for Machine Learning

Prepare for Tutorial of **Week 5 (09.- 13.11.2026)**

Justify all your claims.

Exercise 1 (Symmetric matrices).
Prove the following statements.

a) Any orthogonal, symmetric matrix $A \in \mathbb{R}^{n\times n}$ can only have the eigenvalues -1 or 1.

b) Consider a symmetric matrix $A \in \mathbb{R}^{n\times n}$ and an orthogonal matrix $Q \in \mathbb{R}^{n\times n}$ and let $\tilde{A} = QAQ^T$. Then $\tilde{A}$ is again symmetric and has the same eigenvalues as A.

c) For any matrix $A \in \mathbb{R}^{n\times m}$ with $A^TA = I_m$ the matrix AA^T is the orthogonal projection onto the space range(A).
Hint: To show $\ker(AA^T) = \text{range}(A)^\perp$ *show "$\subseteq$" and "$\supseteq$" separately.*

Exercise 2 (Scalar products and norms).
Consider a normed vector space $(V, \|\cdot\|)$. The parallelogram equality states that the norm is induced by a scalar product $\langle\cdot,\cdot\rangle : V \times V \to \mathbb{R}$ via $\|v\| = \sqrt{\langle v, v\rangle}$ if and only if for all $v, w \in V$ it holds that

$$\|v + w\|^2 + \|v - w\|^2 = 2\left(\|v\|^2 + \|w\|^2\right) \tag{1}$$

a) Prove the forward implication. That is, prove that Eq. (1) holds if the norm is given by a scalar product.

b) Let $p > 0$. Prove that there is a scalar product on $\mathbb{R}^2$ such that the associated norm is given by

$$\|x\| = (|x_1|^p + |x_2|^p)^{1/p}$$

for all $x \in \mathbb{R}^2$ if and only if $p = 2$.

c) Let $(V, \|\cdot\|)$ be a normed vector space, where the norm is given by a scalar product. Prove the Pythagorean theorem $\|u + v\|^2 = \|u\|^2 + \|v\|^2$ for all orthogonal $u, v \in V$.

Exercise 3 (Power Method).
Let $A \in \mathbb{R}^{n\times n}$ be a diagonalizable matrix with one unique largest eigenvalue, that is, $|\lambda_1| > |\lambda_2| > ... > |\lambda_n|$, where λ_i are the eigenvalues. We furthermore assume $\lambda_1 > 0$.
We consider the *power method*, a method to numerically estimate an eigenvector to the largest eigenvalue λ_1. For an arbitrary initial vector $x_0 \in \mathbb{R}^n$ we recursively define

$$x_{k+1} := \frac{Ax_k}{\|Ax_k\|},$$

where $\|\cdot\|$ denotes the Euclidean norm.

a) Use induction to prove that $x_k = \frac{A^k x_0}{\|A^k x_0\|}$.

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b) Consider a basis of eigenvectors $v_1, ..., v_n$, where v_i belongs to λ_i, and the representation

$$x_0 = c_1 v_1 + ... + c_n v_n.$$

Prove that, if $c_1 \neq 0$, the sequence x_k converges to an eigenvector of λ_1 for $k \to \infty$.

For an intuitive illustration of what this method does, see part 3.2 of Exercise 4.

Exercise 4 (Visualization of linear maps).
In this exercise based on a jupyter notebook, you can get an intuition of what linear maps, eigenvectors, diagonalization, the spectral theorem, and the power method do. You may download the files from the website. You work with the file eigen_exercise.ipynb, the file eigen_helpers.py needs to sit in the same folder.
Note that there will be no quiz based on this exercise, but we still recommend you do it as it is very helpful for getting a feel for these concepts.

