Winter term 2026/27
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Assignment 02

Mathematics for Machine Learning

Prepare for Tutorial of **Week 04 (02. - 06.11.2026)**

Justify all your claims.

Exercise 1 (Change of Basis).
Consider the linear map $T \in \mathcal{L}(\mathbb{R}^3, \mathbb{R}^3)$ with $T(x) = (-x_1, x_2, 2x_3)^t$ for $x = (x_1, x_2, x_3)^t \in \mathbb{R}^3$.
Consider the standard basis $\mathcal{B} = \{e_1, e_2, e_3\}$ and the basis $\mathcal{C} = \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$ of $\mathbb{R}^3$.

a) Find the matrix $\mathcal{M}(T, \mathcal{B}, \mathcal{B})$ which corresponds to the linear map T.

b) Find the transformation matrices $\mathcal{M}(\mathrm{Id}, \mathcal{B}, \mathcal{C})$ and $\mathcal{M}(\mathrm{Id}, \mathcal{C}, \mathcal{B})$.

c) Find the matrix $\mathcal{M}(T, \mathcal{C}, \mathcal{C})$.

Solution:

a) Calculate $T(e_1), T(e_2), T(e_3)$ to get the matrix:

$$\mathcal{M}(T, \mathcal{B}, \mathcal{B}) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

b) Read off the first transformation matrix:

$$\mathcal{M}(\mathrm{Id}, \mathcal{C}, \mathcal{B}) = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 1 \\ 3 & 2 & 1 \end{pmatrix}$$

and calculate its inverse to get the second transformation matrix:

$$\mathcal{M}(\mathrm{Id}, \mathcal{B}, \mathcal{C}) = \mathcal{M}(\mathrm{Id}, \mathcal{C}, \mathcal{B})^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & -1 & 1 \\ -1 & 2 & -1 \end{pmatrix}$$

c) Use the result of the lecture and get:

$$\mathcal{M}(T, \mathcal{C}, \mathcal{C}) = \mathcal{M}(\mathrm{Id}, \mathcal{B}, \mathcal{C}) \cdot \mathcal{M}(T, \mathcal{B}, \mathcal{B}) \cdot \mathcal{M}(\mathrm{Id}, \mathcal{C}, \mathcal{B}) = \begin{pmatrix} -1 & 0 & 0 \\ 5 & 3 & 1 \\ -1 & -2 & 0 \end{pmatrix}$$

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Exercise 2 (**Matrices).**

a) Give an example for a 2-by-2 matrix A with $A \neq 0$, but $A^2 = 0$.

b) Consider two square matrices $A, B \in \mathbb{R}^{n\times n}$ with $A \cdot B = I$. Prove that $B \cdot A = I$

c) Consider a matrix $A \in \mathbb{R}^{n\times n}$ with $A^2 = 0$. Show that $A - I$ is invertible.

Solution:

a) For example $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

b) The assumption implies that B is surjective. One way to see this is that $1 = \det(I) = \det(A \cdot B) = \det A \det B$, so $\det B \neq 0$, which is equivalent to B being bijective. Let $v \in \mathbb{R}^n$. Since B is surjective, there exists a $w \in \mathbb{R}^n$ with $v = Bw$. It is

$$(BA)v = (BA)Bw = B(AB)w = Bw = v = Iv\,.$$

Since v was arbitrary, it follows that $BA = I$.

c) It holds $(A - I)^{-1} = -(A + I)$, as we verify by

$$-(A+I)(A-I) = -\left(A^2 - A \cdot I + I \cdot A - I^2\right) = -\underbrace{A^2}_{=0} + \underbrace{A - A}_{=0} + I = I\,.$$

Exercise 3 (**Matrices**).
Consider the differentiation operator $D = \mathrm{d}/\mathrm{d}t \colon \mathbb{R}^{\mathbb{R}} \to \mathbb{R}^{\mathbb{R}}, f \mapsto f'$ on the vector space $\mathbb{R}^{\mathbb{R}}$ of all real functions. Below we give different choices of bases $\mathcal{W}$. For each of them, we consider the corresponding subspace $\mathcal{U} := \operatorname{span}(\mathcal{W})$ and the restricted linear map $D|_{\mathcal{U}} \colon \mathcal{U} \to \mathbb{R}^{\mathbb{R}}$, which is the differentiation operator just applied to vectors in $\mathcal{U}$. Decide whether $\operatorname{range}(D|_{\mathcal{U}}) \subseteq \mathcal{U}$ and if so, state the matrix $\mathcal{M}(D|_{\mathcal{U}}, \mathcal{W}, \mathcal{W})$.

a) $\mathcal{W} = \{e^t, e^{2t}\}$

b) $\mathcal{W} = \{1, t^2, t^4\}$

c) $\mathcal{W} = \{e^t, te^t\}$

d) $\mathcal{W} = \{\sin t, \cos t\}$

Solution: To compute $\mathcal{M}(D|_{\mathcal{U}}, \mathcal{W}, \mathcal{W})$, we have to express each derivative $D|_{\mathcal{U}}(w_i)$ as a linear combination of the basis vectors in $\mathcal{W}$. The first two exercises are proved as an example; for the others only the solution is stated.

a) It is

$$\begin{aligned} D|_{\mathcal{U}}(e^t) &= e^t \\ D|_{\mathcal{U}}(e^{2t}) &= 2 \cdot e^{2t} \end{aligned}$$

and therefore $\mathit{range}(D|_{\mathcal{U}}) = \mathit{span}(e^t, 2 \cdot e^t) = \mathit{span}(\mathcal{W})$. Further we can write

$$\begin{aligned} D|_{\mathcal{U}}(e^t) &= 1 \cdot e^t + 0 \cdot e^{2t} \\ D|_{\mathcal{U}}(e^{2t}) &= 0 \cdot e^t + 2 \cdot e^{2t} \end{aligned}$$

and therefore $\mathcal{M}(D|_{\mathcal{U}}, \mathcal{W}, \mathcal{W}) = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$.

b) It is

$$\begin{aligned} D|_{\mathcal{U}}(1) &= 1 \\ D|_{\mathcal{U}}(t^2) &= 2 \cdot t \\ D|_{\mathcal{U}}(t^4) &= 4 \cdot t^3 \end{aligned}$$

and therefore $\mathit{range}(D|_{\mathcal{U}}) = \mathit{span}(1, t, t^3) \not\subset \mathit{span}(\mathcal{W})$.

c) $\mathcal{M}(D|_{\mathcal{U}}, \mathcal{W}, \mathcal{W}) = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$

d) $\mathcal{M}(D|_{\mathcal{U}}, \mathcal{W}, \mathcal{W}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

Exercise 4 (Eigenvalues).

a) Let $A \in \mathbb{R}^{n\times n}$ with $A^k = 0$ for some $k \in \mathbb{N}$. Prove that, if λ is an eigenvalue of A, then $\lambda = 0$.

b) Let V be a finite-dimensional vector space and $T\colon V \to V$ a linear map such that every $v \in V$ with $v \neq 0$ is an eigenvector of T. Prove that $T = \lambda \,\mathrm{Id}$ for some $\lambda \in \mathbb{R}$.

Solution:

a) Assume that there exists an eigenvalue $\lambda \neq 0$ with eigenvector $v \neq 0$. By assumption it is $A^k = 0$, which in particular implies that $v \in \ker(A^k)$. Applying the eigenvalue equation $Av = \lambda v$ iteratively from the left yields

$$0 = A^k v = \lambda A^{k-1} v = \cdots = \lambda^{k-1} A v = \lambda^k v\,.$$

Since $\lambda \neq 0$ it follows $v = 0$, which contradicts our assumption.

b) Let $v_1, \ldots, v_n$ be a basis of V. By assumption, there exist $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ with $Tv_i = \lambda_i v_i$. We only have to prove that all λ_i are equal, because T is uniquely determined by its values on a basis. Because $v_1 + \cdots + v_n \in V$, the assumption also implies that there exists a $\lambda \in \mathbb{R}$ with $T(v_1 + \cdots + v_n) = \lambda(v_1 + \cdots + v_n)$ and therefore

$$\begin{aligned} & \lambda(v_1 + \cdots + v_n) = T(v_1 + \cdots + v_n) = Tv_1 + \cdots + Tv_n = \lambda_1 v_1 + \cdots + \lambda_n v_n \\ \iff \quad & 0 = (\lambda_1 - \lambda)v_1 + \cdots + (\lambda_n - \lambda)v_n\,. \end{aligned}$$

Since $v_1, \ldots, v_n$ is a basis, it is linearly independent, which implies $\lambda_i - \lambda = 0$ for all $i = 1, \ldots, n$. In particular, this implies that all λ_i are equal, which completes the proof.