

Assignment 02

Mathematics for Machine Learning

Prepare for Tutorial of **Week 04 (02. - 06.11.2026)**

Justify all your claims.

Exercise 1 (Change of Basis).

Consider the linear map $T \in \mathcal{L}(\mathbb{R}^3, \mathbb{R}^3)$ with $T(x) = (-x_1, x_2, 2x_3)^t$ for $x = (x_1, x_2, x_3)^t \in \mathbb{R}^3$.

Consider the standard basis $\mathcal{B} = \{e_1, e_2, e_3\}$ and the basis $\mathcal{C} = \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$ of $\mathbb{R}^3$.

- Find the matrix $\mathcal{M}(T, \mathcal{B}, \mathcal{B})$ which corresponds to the linear map T .
- Find the transformation matrices $\mathcal{M}(\text{Id}, \mathcal{B}, \mathcal{C})$ and $\mathcal{M}(\text{Id}, \mathcal{C}, \mathcal{B})$.
- Find the matrix $\mathcal{M}(T, \mathcal{C}, \mathcal{C})$.

Exercise 2 (Matrices).

- Give an example for a 2-by-2 matrix A with $A \neq 0$, but $A^2 = 0$.
- Consider two square matrices $A, B \in \mathbb{R}^{n \times n}$ with $A \cdot B = I$. Prove that $B \cdot A = I$.
- Consider a matrix $A \in \mathbb{R}^{n \times n}$ with $A^2 = 0$. Show that $A - I$ is invertible.

Exercise 3 (Matrices).

Consider the differentiation operator $D = d/dt: \mathbb{R}^{\mathbb{R}} \rightarrow \mathbb{R}^{\mathbb{R}}, f \mapsto f'$ on the vector space $\mathbb{R}^{\mathbb{R}}$ of all real functions. Below we give different choices of bases $\mathcal{W}$. For each of them, we consider the corresponding subspace $\mathcal{U} := \text{span}(\mathcal{W})$ and the restricted linear map $D|_{\mathcal{U}}: \mathcal{U} \rightarrow \mathbb{R}^{\mathbb{R}}$, which is the differentiation operator just applied to vectors in $\mathcal{U}$. Decide whether $\text{range}(D|_{\mathcal{U}}) \subseteq \mathcal{U}$ and if so, state the matrix $\mathcal{M}(D|_{\mathcal{U}}, \mathcal{W}, \mathcal{W})$.

- $\mathcal{W} = \{e^t, e^{2t}\}$
- $\mathcal{W} = \{1, t^2, t^4\}$
- $\mathcal{W} = \{e^t, te^t\}$
- $\mathcal{W} = \{\sin t, \cos t\}$

Exercise 4 (Eigenvalues).

- Let $A \in \mathbb{R}^{n \times n}$ with $A^k = 0$ for some $k \in \mathbb{N}$. Prove that, if λ is an eigenvalue of A , then $\lambda = 0$.
- Let V be a finite-dimensional vector space and $T: V \rightarrow V$ a linear map such that every $v \in V$ with $v \neq 0$ is an eigenvector of T . Prove that $T = \lambda \text{Id}$ for some $\lambda \in \mathbb{R}$.