Winter term 2026/27
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Assignment 01

Mathematics for Machine Learning

Prepare for Tutorial of **Week 03 (26. - 30.10.2026)**

Justify all your claims.

Exercise 1 (Dictionary Learning).
In this exercise, you will explore the concept of dictionary learning, where the goal is to find a set of non-zero vectors D (a "dictionary") such that data points can be represented as sparse linear combinations of these vectors. For further reading, you can look up "dictionary learning" or "compressed sensing".
Note: For this exercise we consider only linear combinations $\sum_j \lambda_j d_j$ with non-zero scalars λ_j.

a) Consider a vector $v = (2, 1, 1) \in \mathbb{R}^3$ and a dictionary D consisting of the following five vectors:

$$D = \{d_1 = (1, 0, 0), d_2 = (0, 1, 0), d_3 = (0, 0, 1), d_4 = (1, 1, 0), d_5 = (0, 1, 1)\}.$$

Express v as a linear combination of

i) exactly three dictionary vectors.

ii) exactly four dictionary vectors.

iii) all five dictionary vectors.

b) Consider a vector $v \in \mathbb{R}^3$ and a dictionary D consisting of five vectors spanning $\mathbb{R}^3$. Can v always be written in multiple ways as a linear combination of the dictionary vectors? If yes, provide a proof. If no, give a counterexample.

c) Find a dictionary in $\mathbb{R}^3$ with a minimal number of vectors, such that each of the following points can be expressed as a linear combination of exactly two dictionary vectors.

i) $v_1 = (1, 1, 1), \quad v_2 = (2, 1, 1), \quad v_3 = (1, 2, 2)$

ii) $v_1 = (1, 1, 1), \quad v_2 = (2, 1, 0), \quad v_3 = (1, 2, 0)$

Solution:

a) i) $v = 2d_1 + d_2 + d_3$

ii) $v = d_1 - d_2 + d_4 + d_5$

iii) $v = d_1 + d_2 + 2d_3 + d_4 - d_5$

b) Yes, because the dimension of $\mathbb{R}^3$ is three. Therefore a set of 5 different vectors is linear dependent.

c) i) $D = \{d_1 = (1, 0, 0), d_2 = (0, 1, 1)\}$

ii) $D = \{d_1 = (1, 0, 0), d_2 = (0, 1, 1), d_3 = (0, 1, 0)\}$. A dictionary with just two vectors cannot exist because v_1, v_2, v_3 are linearly independent, so their span is 3-dimensional.

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Exercise 2 (Basis and Dimension).
Let V be a finite-dimensional vector space with basis $\mathcal{B} = (v_1, \dots, v_n)$. Consider a vector $v = \sum_{i=1}^n \lambda_i v_i$ with $\lambda_k \neq 0$ for $1 \leq k \leq n$.
Prove that $\tilde{\mathcal{B}} = (v_1, \dots, v_{k-1}, v, v_{k+1}, \dots, v_n)$ is also a basis of V.

Solution:
W.l.o.g. set $k = 1$.

a) **$\tilde{\mathcal{B}}$ spans V:**
By assumption we have $v_1 = \frac{1}{\lambda_1} v - \sum_{i=2}^n \frac{\lambda_i v_i}{\lambda_1}$.
For $w \in V$ we have $w = \sum_{i=1}^n \mu_i v_i = \frac{\mu_1}{\lambda_1} v + \sum_{i=2}^n \left(\mu_i - \frac{\mu_1 \lambda_i}{\lambda_1}\right) v_i$

b) **$\tilde{\mathcal{B}}$ is lin. independent:**
If $0 = \mu v + \sum_{i=2}^n \mu_i v_i$, then $0 = \mu\lambda_1 v_1 + \sum_{i=2}^n (\mu\lambda_i + \mu_i) v_i$.
Since the vectors in $\mathcal{B}$ are lin. independent, we get $0 = \mu\lambda_1 = \mu\lambda_i + \mu_i$ for $i \in \{2, \dots, n\}$.
By assumption it is $\lambda_1 \neq 0$ resulting in $\mu = 0$ and hence also $\mu_i = 0$ for $i \in \{2, \dots, n\}$.

Exercise 3 (Linear Mappings and Vector Spaces).
Let V and W be two vector spaces (either real or complex). Consider the set of linear mappings $\mathcal{L}(V, W)$ with the operations of addition and scalar multiplication defined by

$$(S + T)(v) := Sv + Tv \text{ and } (\lambda T)(v) := \lambda(Tv)$$

for $S, T \in \mathcal{L}(V, W)$ and for all $v \in V$.

a) Verify that $S + T$ and λT are again linear maps in $\mathcal{L}(V, W)$.

b) Prove that $\mathcal{L}(V, W)$ with the above operations is a vector space (again either real or complex).

Solution:

a) For $S, T \in \mathcal{L}(V, W)$, $v, v_1, v_2 \in V$ and $\lambda \in F$ it holds that

$$\begin{aligned}(S + T)(v_1 + v_2) &= S(v_1 + v_2) + T(v_1 + v_2) \\ &= S(v_1) + S(v_2) + T(v_1) + T(v_2) \\ &= S(v_1) + T(v_1) + S(v_2) + T(v_2) \\ &= (S + T)(v_1) + (S + T)(v_2)\end{aligned}$$

and

$$(S + T)(\lambda v) = S(\lambda v) + T(\lambda v) = \lambda S(v) + \lambda T(v) = \lambda(S(v) + T(v)) = \lambda(S + T)(v).$$

Similar for λT.

b) **$(\mathcal{L}(V, W), +)$ is a commutative group:**

- Associativity: For $S, T, L \in \mathcal{L}(V, W)$ and $v \in V$ we have

$$\begin{aligned}((S + T) + L)(v) &= (S + T)(v) + Lv \\ &= Sv + Tv + Lv \\ &= Sv + (T + L)(v) \\ &= (S + (T + L))(v).\end{aligned}$$

- Identity: Let $\mathbf{0}_W \in \mathcal{L}(V,W)$ with $\mathbf{0}_W(v) = 0_W$ for all $v \in V$, where 0_W is the identity element in W. Then for $S \in \mathcal{L}(V,W)$ and $v \in V$ we have

$$(S + \mathbf{0}_W)(v) = Sv + \mathbf{0}_W(v) = Sv + 0_W = Sv$$

and

$$(\mathbf{0}_W + S)(v) = \mathbf{0}_W(v) + Sv = 0_W + Sv = Sv.$$

- Inverse: Let $-S \in \mathcal{L}(V,W)$ with $(-S)(v) = -(Sv)$ (exists because W is a vector space). Then for $S \in \mathcal{L}(V,W)$ and $v \in V$ we have

$$(S + (-S)(v) = Sv + (-S)(v) = Sv - Sv = 0$$

and

$$((-S) + S)(v) = (-S)v + Sv = -Sv + Sv = 0.$$

- Commutativity: $(S+T)(v) = Sv + Tv = Tv + Sv = (T+S)(v)$

Multiplicative identity element:
Let $1 \in F$ be the identity element of the underlying field. Then

$$(1 \cdot S)(w) = 1 \cdot (Sw) = Sw.$$

Distributivity:

$$\begin{aligned}(\lambda(S+T))(v) &= \lambda((S+T)(v)) \\ &= \lambda(Sv + Tv) \\ &= \lambda(Sv) + \lambda(Tv) \\ &= (\lambda S)(v) + (\lambda T)(v) \\ &= (\lambda S + \lambda T)(v)\end{aligned}$$

and

$$((\lambda + \mu)(S))(v) = (\lambda + \mu)(Sv) = \lambda(Sv) + \mu(Sv) = (\lambda S)(v) + (\mu S)(v) = (\lambda S + \mu S)(v).$$

Exercise 4 (Linear Mappings).
Suppose $v_1, \ldots, v_n$ is a list of vectors in some vector space V. Define $T \in \mathcal{L}(\mathbb{R}^n, V)$ by $T(\lambda_1, \ldots, \lambda_n) = \lambda_1 v_1 + \cdots + \lambda_n v_n$.

a) Which property does T need to have such that the following holds: The vectors $v_1, \ldots, v_n$ span the space V if and only if T has this property?

b) Which property does T need to have such that the following holds: The vectors $v_1, \ldots, v_n$ are linearly independent if and only if T has this property?

Solution:

a) $\operatorname{span}(v_1 \ldots v_n) = V$
$\iff \operatorname{range}(T) = V$
$\iff T$ surjective

b) $v_1 \ldots v_n$ lin. independent
$\iff 0 = \sum_{i=1}^n \lambda_i v_i$ implies that $\lambda_1 = \cdots = \lambda_n = 0$
$\iff \ker(T) = \{0\}$
$\iff T$ injective