

Assignment 01

Mathematics for Machine Learning

Prepare for Tutorial of **Week 03 (26. - 30.10.2026)**

Justify all your claims.

Exercise 1 (Dictionary Learning).

In this exercise, you will explore the concept of dictionary learning, where the goal is to find a set of non-zero vectors D (a "dictionary") such that data points can be represented as sparse linear combinations of these vectors. For further reading, you can look up "dictionary learning" or "compressed sensing".

Note: For this exercise we consider only linear combinations $\sum_j \lambda_j d_j$ with non-zero scalars λ_j .

- a) Consider a vector $v = (2, 1, 1) \in \mathbb{R}^3$ and a dictionary D consisting of the following five vectors:

$$D = \{d_1 = (1, 0, 0), d_2 = (0, 1, 0), d_3 = (0, 0, 1), d_4 = (1, 1, 0), d_5 = (0, 1, 1)\}.$$

Express v as a linear combination of

- i) exactly three dictionary vectors.
 - ii) exactly four dictionary vectors.
 - iii) all five dictionary vectors.
- b) Consider a vector $v \in \mathbb{R}^3$ and a dictionary D consisting of five vectors spanning $\mathbb{R}^3$. Can v always be written in multiple ways as a linear combination of the dictionary vectors? If yes, provide a proof. If no, give a counterexample.
- c) Find a dictionary in $\mathbb{R}^3$ with a minimal number of vectors, such that each of the following points can be expressed as a linear combination of exactly two dictionary vectors.
- i) $v_1 = (1, 1, 1), \quad v_2 = (2, 1, 1), \quad v_3 = (1, 2, 2)$
 - ii) $v_1 = (1, 1, 1), \quad v_2 = (2, 1, 0), \quad v_3 = (1, 2, 0)$

Exercise 2 (Basis and Dimension).

Let V be a finite-dimensional vector space with basis $\mathcal{B} = (v_1, \dots, v_n)$. Consider a vector $v = \sum_{i=1}^n \lambda_i v_i$ with $\lambda_k \neq 0$ for $1 \leq k \leq n$.

Prove that $\tilde{\mathcal{B}} = (v_1, \dots, v_{k-1}, v, v_{k+1}, \dots, v_n)$ is also a basis of V .

Exercise 3 (Linear Mappings and Vector Spaces).

Let V and W be two vector spaces (either real or complex). Consider the set of linear mappings $\mathcal{L}(V, W)$ with the operations of addition and scalar multiplication defined by

$$(S + T)(v) := Sv + Tv \text{ and } (\lambda T)(v) := \lambda(Tv)$$

for $S, T \in \mathcal{L}(V, W)$ and for all $v \in V$.

- a) Verify that $S + T$ and λT are again linear maps in $\mathcal{L}(V, W)$.
- b) Prove that $\mathcal{L}(V, W)$ with the above operations is a vector space (again either real or complex).

Exercise 4 (Linear Mappings).

Suppose $v_1, \dots, v_n$ is a list of vectors in some vector space V . Define $T \in \mathcal{L}(\mathbb{R}^n, V)$ by $T(\lambda_1, \dots, \lambda_n) = \lambda_1 v_1 + \dots + \lambda_n v_n$.

- a) Which property does T need to have such that the following holds: The vectors $v_1, \dots, v_n$ span the space V if and only if T has this property?
- b) Which property does T need to have such that the following holds: The vectors $v_1, \dots, v_n$ are linearly independent if and only if T has this property?